

# Markov property and Khovanov-Rozansky homology: Coxeter Case.

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## Abstract

In [2] Khovanov gave a construction of Khovanov-Rozansky homology, new triple-graded link invariant, taking the Hochschild homology of the terms in Rouquier complexes. Its bigraded Euler characteristic provides the Markov trace of type  $A_n$ , which, in its turn, induces the well-known HOMFLY polynomial. Gomi [1, (3.3)] has extended the definition of Markov trace on a Hecke algebra to all Coxeter groups. One can see that Khovanov's and Rouquier's constructions can be directly extended to these groups. We give a detailed proof that the corresponding Euler characteristic provides a Markov trace in the sense of Gomi for any Coxeter group.

Let  $(W, S)$  be a Coxeter system with  $|S| = n < \infty$ . Let  $V$  be the  $\mathbb{C}$ -vector space with the basis  $\{e_s\}_{s \in S}$  carrying the geometric representation of  $W$ . Then we have a natural action of  $W$  on the symmetric algebra  $R = S(V^*)$ , which will be regarded as the graded polynomial ring  $\mathbb{C}[\{x_s\}_{s \in S}]$  with  $\deg x_s = 1$ .

Now we briefly recall the notions of Rouquier complexes needed to give Khovanov's construction of Khovanov-Rozansky homology. Let  $B_W = \langle \sigma_s \rangle_{s \in S}$  be the corresponding to  $W$  braid group. For  $s \in S$  define the graded  $R$ -bimodule  $B_s = R \otimes_{R^s} R$ , where  $R^s$  is the ring of polynomials invariant under  $s$ . Graduation on  $B_s$  is induced by the graduation on  $R$ . To the braid generator  $\sigma_s$  (resp.  $\sigma_s^{-1}$ ) Rouquier associates cochain complexes of graded  $R$ -bimodules:

$$F(\sigma_s) : 0 \rightarrow B_s \xrightarrow{m} R \rightarrow 0 \quad \text{resp.} \quad F(\sigma_s^{-1}) : 0 \rightarrow R \xrightarrow{\eta} B_s(1) \rightarrow 0$$

where  $B_s(1)$  means the shift in the graduation by 1,  $m(a \otimes b) = ab$  and  $\eta(a) = a \otimes x_s + ax_s \otimes 1$ . Bimodules  $B_s$  and  $B_s(1)$  are placed in cohomological degree 0. Rouquier [4, 9.] proves the following

**Theorem 1 (Rouquier)** *Map  $\sigma_s^{\pm 1} \rightarrow F(\sigma_s^{\pm 1}) \otimes_R -$  induces a well-defined action of the braid group  $B_W$  on the category of cochain complexes of graded  $R$ -bimodules up to homotopy.*

The above theorem allows to define the *Rouquier complex*  $F(\sigma)$  of  $\sigma \in B_W$  as the image of the complex  $0 \rightarrow R \rightarrow 0$  by the action of  $\sigma$ .

Recall that the *Hochschild homology*  $\mathrm{HH}(R, M)$  of an  $R$ -bimodule  $M$  is defined as the direct sum of  $\mathrm{Tor}_i^{R \otimes R}(R, M)$ . For  $\sigma \in B_W$  Khovanov takes the Hochschild homology of each term in the Rouquier complex  $F(\sigma)$  and gets a complex of bigraded spaces

$$\dots \xrightarrow{\mathrm{HH}(\partial)} \mathrm{HH}(R, F^j(\sigma)) \xrightarrow{\mathrm{HH}(\partial)} \mathrm{HH}(R, F^{j+1}(\sigma)) \xrightarrow{\mathrm{HH}(\partial)} \dots$$

The homology of the above complex is called *Khovanov-Rozansky homology*. When  $W$  is the symmetric group Khovanov proves the following

**Theorem 2 (Khovanov)** *The triple-graded dimension of Khovanov-Rozansky homology is an invariant of oriented links. Its bigraded Euler characteristic provides the HOMFLY 2-variable polynomial.*

Let  $\mathcal{H}_W := \mathbb{C}(q)B_W / \langle (\sigma_s - q)(\sigma_s + 1) \rangle$  be the Hecke algebra of  $W$ . We will explain how the Euler characteristic of Khovanov-Rozansky homology provides a trace function on  $\mathcal{H}_W$ . For  $\sigma \in B_W$  define an element  $\langle \sigma \rangle_V \in \mathbb{C}(q, t)$  as

$$\langle \sigma \rangle_V = \left( \frac{1-q}{1+tq} \right)^n \cdot \sum_{i,j,k} (-1)^i \dim_{\mathbb{C}} \mathrm{HH}_j(R, F^i(\sigma))_k \cdot q^k \cdot t^j,$$

where the index  $i$  corresponds to the cohomological grading of the Rouquier complex,  $j$  - to the Hochschild homology grading and  $k$  - to the polynomial grading of  $R$ -modules  $\mathrm{HH}_j(R, F^i(\sigma))$ . By linearity we get a  $\mathbb{C}(q)$ -linear map  $\langle \cdot \rangle_V : \mathbb{C}(q)B_W \rightarrow \mathbb{C}(q, t)$ . The following proposition is well-known (see e.g. [3, (4.4.6)]):

**Proposition 1** *The map  $\langle \cdot \rangle_V$  has the following properties:*

- $\langle 1 \rangle_V = 1$  (Normalizing property)
- $\langle ab \rangle_V = \langle ba \rangle_V$  for any  $a, b \in \mathbb{C}(q)B_W$  (Trace property)
- For any  $a \in \mathbb{C}(q)B_W$  and  $s \in S$  we have  $\langle a\sigma_s \rangle_V = q \langle a\sigma_s^{-1} \rangle_V + (q-1) \langle a \rangle_V$  (Hecke property)

This proposition implies that the map  $\langle \cdot \rangle_V$  can be factorized through the Hecke algebra  $\mathcal{H}_W$ . It induces a trace  $\tau_{kr}$  on  $\mathcal{H}_W$  called *Khovanov-Rozansky trace*.

Let  $\mathbf{T}_s$  be the image of  $\sigma_s$  in  $\mathcal{H}_W$ . Recall, that Gomi defines a *Markov trace* with a parameter  $z \in \mathbb{C}(q, t)$  on  $\mathcal{H}_W$  as a  $\mathbb{C}(q)$ -linear function  $\tau$  with the properties

- $\tau(1) = 1$ ,  $\tau(\mathbf{a}\mathbf{b}) = \tau(\mathbf{b}\mathbf{a})$  for any  $\mathbf{a}, \mathbf{b} \in \mathcal{H}_W$
- $\tau(\mathbf{h}\mathbf{T}_s) = z\tau(\mathbf{h})$  for any  $s \in S$  and any  $\mathbf{h}$  from the parabolic subalgebra  $\mathcal{H}_{W, S \setminus \{s\}}$  of  $\mathcal{H}_W$  generated by the set  $\{\mathbf{T}_{s'}\}_{s' \in S \setminus \{s\}}$ . This property is called *Markov property*.

The result of this article is the following theorem:

**Theorem 3** *The trace  $\tau_{kr}$  is a Markov trace with the parameter  $z = \frac{tq(q-1)}{tq+1}$ .*

**Proof.** The idea of the proof is based on the proof of [2, Th. 1] where Koszul complexes are used. Let  $e_1, \dots, e_n$  be any basis of the vector space  $V$ . We denote the dual  $e_i^*$  by  $x_i \in R$ . We will use the following notations:

$$\begin{aligned} R_x^n &:= \mathbb{C}[x_1, \dots, x_n], & R_y^n &:= \mathbb{C}[y_1, \dots, y_n], & R_z^n &:= \mathbb{C}[z_1, \dots, z_n], & R_{x,y,z}^n &:= R_x^n \otimes_{\mathbb{C}} R_y^n \otimes_{\mathbb{C}} R_z^n, \\ R_{x,y}^n &:= R_x^n \otimes_{\mathbb{C}} R_y^n, & R_{x,z}^n &:= R_x^n \otimes_{\mathbb{C}} R_z^n, & R_{y,z}^n &:= R_y^n \otimes_{\mathbb{C}} R_z^n. \end{aligned}$$

The action of  $W$  on  $R_y^n, R_z^n$  is exactly the same as on  $R = R_x^n$  and is directly induced on  $R_{x,y}^n, R_{x,z}^n, R_{y,z}^n, R_{x,y,z}^n$ .

Let  $A$  be a graded ring and  $a_1, \dots, a_k$  some homogeneous elements in  $A$ . We denote by  $(a_1, \dots, a_k)_A$  the corresponding *Koszul complex* which is defined as the tensor product over  $A$  of small complexes

$$0 \rightarrow A(-\deg a_i) \xrightarrow{\times a_i} A \rightarrow 0, \quad \text{where } \times a_i \text{ means the multiplication by } a_i.$$

**Fact 1** Let  $(a_1, \dots, a_k)_A$  be a Koszul complex and  $\lambda \in A$  be an element such that  $\lambda a_i$  is homogeneous of the same degree as  $a_j$  for some  $i \neq j$ . Then we have the following isomorphism of complexes:

$$(a_1, \dots, a_i, \dots, a_j \dots, a_k)_A \simeq (a_1, \dots, a_i, \dots, a_j + \lambda a_i \dots, a_k)_A.$$

**Fact 2** Let  $A, B$  be some  $\mathbb{C}$ -algebras,  $K^\bullet$  some complex of  $A \otimes_{\mathbb{C}} B$ -modules and  $L^\bullet$  a complex of  $A$ -modules. Then there is the obvious isomorphism of complexes  $K^\bullet \otimes_A L^\bullet \simeq K^\bullet \otimes_{A \otimes B} (L^\bullet \otimes_{\mathbb{C}} B)$ .

Proposition 1, the definition and the  $\mathbb{C}(q)$ -linearity of  $\tau_{kr}$  imply that in order to prove the theorem it is enough to show that  $\langle b\sigma_s \rangle_V = z \langle b \rangle_V$  for any  $s \in S$  and  $b$  being a product of some generators  $\sigma_t, t \neq s$  of the braid group  $B_W$ .

For a bigraded space  $M$  with finite-dimensional homogeneous components we denote its bigraded dimension by  $\dim_{\text{gr}}_{q,t} M$ . For a finitely generated graded  $R$ -bimodule  $N$  we denote by  $\spadesuit_{q,t}(N)$  the bigraded dimension of its Hochschild homology  $\dim_{\text{gr}}_{q,t} \text{HH}(R, N) = \sum_{j,k} \dim_{\mathbb{C}} \text{HH}_j(R, N)_k \cdot q^k \cdot t^j$ . Recall that by definition

$$\begin{aligned} \left( \frac{1+tq}{1-q} \right)^n \langle b\sigma_s \rangle_V &= \sum_i (-1)^i \spadesuit_{q,t}(F^i(b\sigma_s)) = \sum_i (-1)^i \spadesuit_{q,t}(F^i(b) \otimes_R B_s) - \sum_i (-1)^i \spadesuit_{q,t}(F^i(b)) \\ \left( \frac{1+tq}{1-q} \right)^n z \langle b \rangle_V &= \frac{tq(q-1)}{tq+1} \sum_i (-1)^i \spadesuit_{q,t}(F^i(b)). \end{aligned}$$

Thus it is enough to prove the following equality:  $(tq+1)\spadesuit_{q,t}(\Theta_b \otimes_R B_s) = (tq^2+1)\spadesuit_{q,t}(\Theta_b)$ , where the bimodule  $\Theta_b$  is a tensor product over  $R$  of some  $B_t$ 's ( $t \neq s$ ) since every  $F^i(b)$  is a direct sum of such bimodules.

Let us look at our vector space  $V$ . Since  $\dim_{\mathbb{C}} V = n > n-1 = |S \setminus \{s\}|$  there exists a vector  $v \in V^*$  invariant under the action of any  $s' \in S, s' \neq s$ . We can assume that  $v = x_n$  and  $x_1, \dots, x_{n-1} \in V^s$ . Let  $u_x = x_n - w_x$  ( $w_x \in V^{*s}$ ) be an element of  $V^*$  dual to the root of  $s$  (vectors  $u_y = y_n - w_y$  and  $u_z = z_n - w_z$  will be the corresponding copies of  $u_x$  in  $R_y^n$  and  $R_z^n$ ).

**Lemma 1** There is an isomorphism of  $R_{x,y}^n$ -modules  $\Theta_b \simeq \Theta'_b \otimes_{\mathbb{C}} \mathbb{C}[x_n]$ , where  $\Theta'_b$  is some  $R_{x,y}^{n-1}$ -module,  $y_n$  acts on  $\mathbb{C}[x_n]$  by multiplication by  $x_n$ .

**Proof.** Easy consequence from the fact that  $\Theta_b$  is a tensor product of some  $B_t$ 's.  $\square$

**Proposition 2** Let  $\widetilde{\Theta}'_b$  be a free  $R_{x,y}^{n-1}$ -resolution of  $\Theta'_b$ . Then the following complex  $K_{b,s}$  is a free  $R_{x,y,z}^n$ -resolution of  $\Theta_b \otimes_{R_y^n} B_s$ :

$$\left( \widetilde{\Theta}'_b \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R_z^n \right) \otimes_{R_{x,y,z}^n} (x_n - y_n, y_1 - z_1, \dots, y_{n-1} - z_{n-1}, u_y^2 - u_z^2)_{R_{x,y,z}^n}.$$

**Proof.** Indeed, the above complex is isomorphic to

$$\begin{aligned} & \left[ \left( \widetilde{\Theta}'_b \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \right) \otimes_{R_{x,y}^n} (x_n - y_n)_{R_{x,y}^n} \right] \otimes_{R_{x,y}^n} (y_1 - z_1, \dots, y_{n-1} - z_{n-1}, u_y^2 - u_z^2)_{R_{x,y,z}^n} \\ & \simeq \left[ \widetilde{\Theta}'_b \otimes_{R_{x,y}^{n-1}} (x_n - y_n)_{R_{x,y}^n} \right] \otimes_{R_y^n} (y_1 - z_1, \dots, y_{n-1} - z_{n-1}, u_y^2 - u_z^2)_{R_{y,z}^n} \end{aligned}$$

Since the complex  $\widetilde{\Theta}'_b \otimes_{R_{x,y}^{n-1}} (x_n - y_n)_{R_{x,y}^n}$  is isomorphic to  $\widetilde{\Theta}'_b \otimes_{\mathbb{C}} (x_n - y_n)_{\mathbb{C}[x_n, y_n]}$ , it is a free  $R_{x,y}^n$ -resolution of  $\Theta_b$ . At the same time the complex  $(y_1 - z_1, \dots, y_{n-1} - z_{n-1}, u_y^2 - u_z^2)_{R_{y,z}^n}$  considered as a complex

of  $R_y^n$ -modules is homotopy equivalent to the free  $R_y^n$ -module  $B_s$ . This proves that their tensor product over  $R_y^n$  is a free  $R_{x,y,z}^n$ -resolution of  $\Theta_b \otimes_{R_y^n} B_s$ .  $\square$

In order to calculate  $\mathrm{HH}(R, \Theta_b \otimes_R B_s)$  we take the tensor product over  $R_{x,z}^n$  of  $K_{b,s}$  and the free  $R_{x,z}^n$ -resolution of  $R$  represented by the Koszul complex  $(x_1 - z_1, \dots, x_n - z_n)_{R_{x,z}^n}$ . This tensor product equals

$$K_{b,s} \otimes_{R_{x,y,z}^n} (x_1 - z_1, \dots, x_n - z_n)_{R_{x,y,z}^n}, \quad \text{which, in its turn, is equal to}$$

$$\begin{aligned} & \left( \widetilde{\Theta}'_b \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R_z^n \right) \otimes_{R_{x,y,z}^n} (x_n - y_n, y_1 - z_1, \dots, y_{n-1} - z_{n-1}, u_y^2 - u_z^2, x_1 - z_1, \dots, x_n - z_n)_{R_{x,y,z}^n} \\ & \simeq \left( \widetilde{\Theta}'_b \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R_z^n \right) \otimes_{R_{x,y,z}^n} (x_n - y_n, y_1 - z_1, \dots, y_{n-1} - z_{n-1}, 0, x_1 - z_1, \dots, x_n - z_n)_{R_{x,y,z}^n}, \end{aligned}$$

where the element  $u_y^2 - u_z^2 = (y_n - z_n - w_y + w_z)(y_n + z_n - w_y - w_z)$  of homogeneous degree 2 was "killed" by elements  $x_n - y_n, x_n - z_n, y_1 - z_1, \dots, y_{n-1} - z_{n-1}$ . Denoting the homology of the complex

$$\left( \widetilde{\Theta}'_b \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R_z^n \right) \otimes_{R_{x,y,z}^n} (x_n - y_n, y_1 - z_1, \dots, y_{n-1} - z_{n-1}, x_1 - z_1, \dots, x_n - z_n)_{R_{x,y,z}^n}$$

by  $H_{b,s}^\bullet$  (which is double-graded) we have  $\spadesuit_{q,t}(\Theta_b \otimes_R B_s) = (1 + tq^2) \dim_{\mathrm{gr}_{q,t}} H_{b,s}^\bullet$ .

The proof that  $\spadesuit_{q,t}(\Theta_b) = (1 + tq) \dim_{\mathrm{gr}_{q,t}} H_{b,s}^\bullet$  is completely analogous. First we make a formal note that the complex  $L_{b,s}$ :

$$\left( \widetilde{\Theta}'_b \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R_z^n \right) \otimes_{R_{x,y,z}^n} (x_n - y_n, y_1 - z_1, \dots, y_{n-1} - z_{n-1}, y_n - z_n)_{R_{x,y,z}^n}$$

is a free  $R_{x,y,z}^n$ -resolution of  $\Theta_b$ . Then we take the tensor product over  $R_{x,z}^n$  of  $L_{b,s}$  and the Koszul complex  $(x_1 - z_1, \dots, x_n - z_n)_{R_{x,z}^n}$ . And we get

$$\begin{aligned} & \left( \widetilde{\Theta}'_b \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R_z^n \right) \otimes_{R_{x,y,z}^n} (x_n - y_n, y_1 - z_1, \dots, y_{n-1} - z_{n-1}, y_n - z_n, x_1 - z_1, \dots, x_n - z_n)_{R_{x,y,z}^n} \\ & \simeq \left( \widetilde{\Theta}'_b \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R_z^n \right) \otimes_{R_{x,y,z}^n} (x_n - y_n, y_1 - z_1, \dots, y_{n-1} - z_{n-1}, 0, x_1 - z_1, \dots, x_n - z_n)_{R_{x,y,z}^n}, \end{aligned}$$

where the element  $y_n - z_n$  of homogeneous degree 1 was "killed" by elements  $x_n - y_n$  and  $x_n - z_n$ . This ends the proof.  $\square$

**Remark 1** *Raphaël Rouquier told me in private communication that he has a "categorified" version of theorem 3 which will be published later.*

## References

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